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COMMENT

# Monopole scattering spectrum from geometric quantisation

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**Abstract.** The bound-state spectrum of a spinless particle in a Euclidean Taub-NUT metric with negative mass parameter (which also describes asymptotic monopole scattering) is derived from geometric quantisation.

The scattering problem of slow Bogomolny-Prasad-Sommerfield (BPS) monopoles leads, for large-separations, to studying the geodesics in Euclidean Taub-NUT space,

$$\mathrm{d}s^2\left(1+\frac{4m}{r}\right)[\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)]+\left(1+\frac{4m}{r}\right)^{-1}(\mathrm{d}\psi+4m\cos\theta\,\mathrm{d}\phi)^2 \qquad (1)$$

with mass parameter $m=-\frac{1}{2}$ [1]. To the two cyclic variables $\psi$ and $t$ are associated the conserved quantities $q=(1+4m/r)^{-1}(\psi^3+4m\cos\theta\phi^\circ)$ and $E=\frac{1}{2}(1+4m/r)(\boldsymbol{r}^{\circ 2}+q^2)$, interpreted as relative electric charge and energy, respectively. Angular momentum,

$$\boldsymbol{J}=\boldsymbol{r}\times\boldsymbol{p}+(4mq)\boldsymbol{r}/r \qquad \text{where} \qquad \boldsymbol{p}=(1+4m/r)\boldsymbol{r}^\circ \qquad (2)$$

is also conserved. The clue to finding the classical as well as the quantum solutions has been the discovery [1] of a conserved Runge-Lenz-type vector,

$$\boldsymbol{K}=\boldsymbol{p}\times\boldsymbol{J}-4m(E-q^2)\boldsymbol{r}/r. \qquad (3)$$

Indeed, using the crucial relations

$$\boldsymbol{K}^2=(2E-q^2)[\boldsymbol{J}^2-(4mq)^2]+4mq^2(E-q^2)^2 \qquad \text{and} \qquad \boldsymbol{K}\cdot\boldsymbol{J}=-(4m)^2q(E-q^2) \qquad (4)$$

a simple calculation shows that the trajectories are conic sections, namely ellipses, parabolas or hyperbolae depending on the energy being smaller, equal to or larger than $q^2/2$ [1, 2]. (Notice that $E<q^2/2$ is only possible for $m<0$.)

Here we shall only be concerned with the bound states. The spectrum has been found in a variety of ways: either by solving the Schrödinger equation [1] or by relating the problem to a harmonic oscillator [3] or finding a spectrum generating o(2, 1) algebra [4] or by supersymmetric WKB [4, 5], or by applying Pauli's algebraic method [2, 6]. The relative charge becomes quantised, $q=s/4m$, $s=0, \pm\frac{1}{2}, \pm1, \ldots$ and the spectrum is

$$E=(4m)^{-2}(n^2-s^2)^{1/2}[n-(n^2-s^2)^{1/2}] \qquad n=|s|+1, |s|+2, \ldots \qquad (5)$$

with degeneracy $(n^2-s^2)$.

In this note we give yet another method, namely by following the recipe of geometric quantisation [7-11]. For each fixed value $E < q^2/2$, let us consider in fact the rescaled Runge-Lenz vector

$$\boldsymbol{M} = (q^2 - 2E)^{-1/2}\, \boldsymbol{K}. \tag{6}$$

The relations (4) then become

$$\boldsymbol{M}^2 + \boldsymbol{J}^2 = (4m)^2[q^2 + (n/4m)^2] \qquad \text{and} \qquad \boldsymbol{M}\cdot\boldsymbol{J} = -4mqn \tag{7}$$

where we have introduced the (so far real) number

$$n = 4m(E - q^2)/(q^2 - 2E)^{1/2}. \tag{8}$$

It is now convenient to introduce the two vectors

$$\boldsymbol{A}_\pm = (\boldsymbol{M} \pm \boldsymbol{J})/2. \tag{9}$$

By equation (7) both of these vectors have constant length, namely

$$\rho_\pm = |\boldsymbol{A}_\pm| = (n \pm s)/2 \tag{10}$$

where we have introduced the notation $|4mq| = s$. So $\boldsymbol{A}_\pm$ describe two 2-spheres. Gibbons and Ruback [3] demonstrate that $\boldsymbol{A}_+$ and $\boldsymbol{A}_-$ map $M_E$, the space of motions with constant charge and energy symplectomorphically on $(\boldsymbol{S}^2)_+ \times (\boldsymbol{S}^2)_-$, the product of two 2-spheres of radius $\rho_+$ and $\rho_-$, endowed with the sum $\Omega_+ + \Omega_-$ of the canonical sympletic structures of the two 2-spheres. Geometric quantisation requires now [7, 8] that both radii be half-integers,

$$2\rho_\pm = n_\pm \qquad \text{for suitable positive integers } n_\pm. \tag{11}$$

By equation (10) this requires $n \pm s = n_\pm$, proving that both $n = (n_+ + n_-)/2$ and $s = (n_+ - n_-)/2$ are half-integers; $n$ and $s$ are furthermore simultaneously integers or half-integers. Equation (8) can be solved then for $E$ to yield the correct spectrum (5).

The above method is, in fact, just a geometric version of Pauli's procedure: the Poisson brackets of angular momentum $\boldsymbol{J}$ and the rescaled Runge-Lenz vector $\boldsymbol{M}$ are in fact

$$\{J_i, J_k\}\varepsilon_{ikn}J^n \qquad \{J_i, M_k\} = \varepsilon_{ikn}M^n \qquad \{M_i, M_k\} = \varepsilon_{ikn}J^n \tag{12}$$

so they form an o(4) algebra [2, 3]. The generators $\boldsymbol{A}_\pm$ just decompose this o(4) into the sum of two independent o(3). The vectors $\boldsymbol{A}_\pm$ become operators under quantisation, with still close to an o(4) algebra; and those states with fixed charge $q = s/4m$ and energy $E$ carry an irreducible representation space of $\boldsymbol{A}_\pm$. The degeneracy of the energy levels is the dimension of this representation space.

This representation space can be constructed out of polarised sections a suitable line bundle $L$ over $M_E \approx (\boldsymbol{S}^2)_+ \times (\boldsymbol{S}^2)_-$ [7-11]. The line bundle $L$ is the tensor product $L_1 \otimes L_2$ of those over the two independent spheres. For each of the spheres, $L_i$ is itself the tensor product of the pre-quantum line bundle (which only exists for half-integer radii $m/2$) with the bundle of half forms.

It is well known (see, e.g., [11]) that over the 2-sphere the half-form bundle has Chern class $-1$. Consequently, for a sphere of radius $m/2$, the representation space is $(m+1) - 1 = m$ dimensional. Explicitly, for a sphere of radius $m/2$, $z = (m/2)\exp(\mathrm{i}\phi)\tan(\theta/2)$ is a complex coordinate. Choosing the antiholomorphic polarisation, any wavefunction is a linear combination of

$$\Psi^k = \frac{z^k}{(1 + z\bar{z})^{m/2}} \left(\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + z\bar{z})^2}\right)^{1/2} \qquad k = 0, m-1. \tag{13}$$

To sum up, the degeneracy of the $n$th energy level (5) of the Taub-NUT problem is $(2\rho_+)(2\rho_-) = (n+s)(n-s) = n^2 - s^2$. The degeneracy is identical to that found in [11] for a system consisting of a Dirac monopole+Coulomb+suitably chosen $1/r^2$ potential. This is not a coincidence: the two systems are 'hiddenly' symplectomorphic [12]. The spectra are different because the relation of the two systems is complicated. They are, rather, the quantum numbers $n$ and $s$ which are the same.

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